

Embedding ergodic actions of compact quantum groups on C^* -algebras into quotient spaces

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Abstract

The notion of compact quantum subgroup is revisited and an alternative definition is given. Induced representations are considered and a Frobenius reciprocity theorem is obtained. A relationship between ergodic actions of compact quantum groups on C^* -algebras and topological transitivity is investigated. A sufficient condition for embedding such actions in quantum quotient spaces is obtained.

1 Introduction

Consider an ergodic action of a compact group G by automorphisms on the C^* -algebra $\mathcal{C}(X)$ of continuous functions on the compact space X . It is known that this action arises from a transitive right action of G on X by homeomorphisms. Therefore the stabilizer of a point of X is a closed subgroup K of G , and X can be identified with the right coset space $K \backslash G$ as a G -space.

The aim of this note is to understand an appropriate generalization of the above property to ergodic actions of compact quantum groups on noncommutative C^* -algebras. Our interest in this problem arises from the program of formulating an abstract duality theory for compact quantum groups where such a generalization is needed, [3], [6].

The relationship between topological transitivity and ergodicity in the case where G is still a classical compact group but acting on a noncommutative C^* -algebra \mathcal{C} , has been investigated by Longo and Peligrad in [4]. They proved that ergodicity is equivalent to the lack of proper closed G -invariant left ideals in \mathcal{C} .

A general theory of ergodic actions of compact matrix pseudogroups on C^* -algebras has been studied by Boca [2], who proved, among other things, that the spectral subspaces of the action are finite dimensional.

The notion of quantum subgroup and quantum quotient space was first introduced by Podleś in [7] for compact matrix pseudogroups, who computed all the subgroups and quotient spaces of the quantum $SU(2)$ and $SO(3)$ groups.

Later Wang [9], while studying ergodic actions of the universal quantum groups on C^* -algebras, proved that, as in the classical case, actions of compact

quantum groups on quotient spaces are ergodic. But he also showed that not all ergodic actions are of this form. More precisely, he found an example of a compact quantum group acting ergodically on a (even commutative) C^* -algebra, which is not a quotient action by a point stabilizer subgroup.

Wang's example suggests that the desired identification $X = K \backslash G$ should be relaxed, in the quantum case, to the possibility of finding a faithful inclusion of $\mathcal{C}(X)$ in the quantum quotient space $\mathcal{C}(K \backslash G)$.

In the first sections we revisit the notion of compact subgroup and quotient space of a compact quantum group. In Sect. 2 we define a compact quantum group $K = (\mathcal{A}', \Delta')$ to be a subgroup of a compact quantum group $G = (\mathcal{A}, \Delta)$ if there exists a surjective $*$ -homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{A}'$ intertwining the co-products. This definition reduces to Podleś's definition for compact matrix pseudogroups. We give an alternative definition, based on the property that a closed ideal \mathcal{I} of \mathcal{A} should satisfy in order that \mathcal{A}/\mathcal{I} become a subgroup of G through the quotient map. More precisely we show that a necessary and sufficient condition is that \mathcal{I} be a closed coideal of \mathcal{A} (Theorem 2.6).

In Sect. 3 we show that the smooth part of a coideal is not dense in the ideal, although the representation category of a subgroup K is completely determined by the smooth part of the coideal. We show that one can always replace a subgroup K with another subgroup K_{\max} for which the smooth part of the coideal is dense, and with the same representation category. We also show that the equivalence classes of subgroups for which their smooth subalgebras are isomorphic as Hopf $*$ -algebras (equivalent subgroups), are in bijective correspondence with the Hilbert space C^* -subcategories with tensor products, subobjects and direct sums, containing $\text{Rep}(G)$, the representation category of G .

We consider the notion of representation induced by a representation of a subgroup K to the whole group G and we show a Frobenius reciprocity theorem at an algebraic level (Sect. 5).

In the next section we generalize Longo and Peligrad theorem to quantum groups: we show that an action of a compact quantum group $G = (\mathcal{A}, \Delta)$ on a unital C^* -algebra \mathcal{C} is ergodic if and only if \mathcal{C} has no proper closed G -invariant left ideal \mathcal{I} for which the left ideal generated by the image of \mathcal{I} under the action is dense. In the case where the action and the Haar measure are faithful, we show that we can drop the density assumption. We give more equivalent properties based on the lack of certain hereditary G -invariant C^* -subalgebras and certain open projections of \mathcal{C} .

In the last section we look for a necessary and sufficient condition in order that an ergodic G -space \mathcal{C} be embeddable in a quantum quotient space by a subgroup. The idea is the following. If such an embedding were possible, then, assuming for simplicity that the acting quantum group have an everywhere defined counit, the restriction of that counit to the quotient space should induce a $*$ -character on \mathcal{C} . Thus the existence of a $*$ -character is a necessary condition in this case. On the other hand it is not difficult to check that if G is a classical group, acting ergodically on \mathcal{C} , the existence of a $*$ -character on \mathcal{C} actually forces \mathcal{C} to be commutative (see Prop. 7.1), and therefore a quotient G -space. Assuming then the existence of a $*$ -character χ on \mathcal{C} , we define the subgroup G_χ

stabilizing χ and we show that if the action is ergodic and faithful and if G has faithful Haar measure, then \mathcal{C} can be embedded faithfully, with its G -action, into the quantum quotient space $G_\chi \backslash G$.

2 Quantum subgroups and closed coideals

We start defining the notion of compact quantum subgroup of a compact quantum group.

Let $G = (\mathcal{A}, \Delta)$ be a separable compact quantum group in the sense of Woronowicz [12]. Recall that this is a pair of a separable unital C^* -algebra \mathcal{A} and a unital $*$ -algebra homomorphism $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that

- a) $\Delta \otimes \iota \circ \Delta = \iota \otimes \Delta \circ \Delta$, with ι the identity map on \mathcal{A} ,
- b) the sets $\{b \otimes I \Delta(c), b, c \in \mathcal{A}\}$ and $\{I \otimes b \Delta(c), b, c \in \mathcal{A}\}$ both span dense subspaces of $\mathcal{A} \otimes \mathcal{A}$.

The map Δ is usually called the coproduct of \mathcal{A} and property a) is referred to as *coassociativity* of Δ .

Let $G = (\mathcal{A}, \Delta)$ be a compact quantum group, and $\theta(\mathcal{A}, \mathcal{A}) : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ the automorphism which exchanges the order in the tensor product. Then compact quantum group $G_\circ := (\mathcal{A}, \theta(\mathcal{A}, \mathcal{A}) \circ \Delta)$ will be referred to as the group *opposite* to G .

2.1 Definition A pair $K = (\mathcal{A}', \Delta')$ consituted by a unital C^* -algebra \mathcal{A}' and a unital $*$ -homomorphism $\Delta' : \mathcal{A}' \rightarrow \mathcal{A}' \otimes \mathcal{A}'$ is a *compact quantum subgroup* of $G = (\mathcal{A}, \Delta)$ if there exists a unital $*$ -epimorphism $\pi : \mathcal{A} \rightarrow \mathcal{A}'$ such that $\pi \otimes \pi \circ \Delta = \Delta' \circ \pi$.

2.2 Proposition A compact quantum subgroup of a separable compact quantum group is a separable compact quantum group.

Proof Since π is surjective, \mathcal{A}' must be separable as well. Furthermore Δ' is coassociative, as, for $a' = \pi(a) \in \mathcal{A}'$,

$$\begin{aligned} \Delta' \otimes \iota \circ \Delta'(a') &= \Delta' \otimes \iota(\Delta'(\pi(a))) = \\ \Delta' \otimes \iota(\pi \otimes \pi(\Delta(a))) &= \Delta' \circ \pi \otimes \pi(\Delta(a)) = \\ \pi \otimes \pi \otimes \pi(\Delta \otimes \iota \circ \Delta(a)) &= \pi \otimes \pi \otimes \pi(\iota \otimes \Delta \circ \Delta(a)) = \iota \otimes \Delta' \circ \Delta'(a'). \end{aligned}$$

Furthermore the same intertwining relation between coproducts shows that property b) holds for K as well. In fact, for example, for $b' = \pi(b), c' = \pi(c) \in \mathcal{A}'$, elements of the form

$$I \otimes b' \Delta'(c') = I \otimes \pi(b) \Delta' \circ \pi(c) = \pi \otimes \pi(I \otimes b \Delta(c))$$

span a dense subspace of $\pi \otimes \pi(\mathcal{A} \otimes \mathcal{A}) = \mathcal{A}' \otimes \mathcal{A}'$. Thus K is a separable compact quantum group.

One can easily recognize that in the case of compact matrix pseudogroups, this definition agrees with Podles definition [7].

Remark 1 If the algebra \mathcal{A} is commutative, then \mathcal{A}' is commutative as well, since π is surjective. Let G and K be the spectra of \mathcal{A} and \mathcal{A}' respectively, which must be compact groups. The epimorphism π then defines an injective continuous map $K \rightarrow G$, and therefore an identification of K with a closed subgroup of G .

Remark 2 It is known that the Haar measure h of a compact quantum group $G = (\mathcal{A}, \Delta)$ is faithful if and only if the corresponding GNS representation π_h is faithful [10]. Consider the compact quantum group $G_h = (\pi_h(\mathcal{A}), \Delta_h)$ defined in [10]. The surjective map $\pi_h : \mathcal{A} \rightarrow \pi_h(\mathcal{A})$ satisfies the required intertwining relation between Δ and Δ_h . Therefore in the case where h is not faithful, according to the previous definition, G_h should be regarded as a *proper* subgroup of G !

If $K = (\mathcal{A}', \Delta')$ is a compact quantum subgroup of $G = (\mathcal{A}, \Delta)$, the C^* -algebra structure of \mathcal{A}' is precisely the C^* -structure of the quotient C^* -algebra $\mathcal{A}/\ker \pi$. Thus the coproduct of \mathcal{A}' can be pulled back to a coproduct on $\mathcal{A}/\ker \pi$ making it into a compact quantum subgroup of G via the quotient map. What properties a closed ideal \mathcal{J} of \mathcal{A} should then satisfy in order that \mathcal{A}/\mathcal{J} become a compact quantum subgroup of G through the quotient map $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$?

Before discussing this problem, we anticipate the following well known fact. If \mathcal{A} is any C^* -algebra faithfully represented on a Hilbert space \mathcal{H} , by $\iota_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ we shall denote the defining representation of \mathcal{A} .

2.3 Lemma *Let \mathcal{A} and \mathcal{B} be unital C^* -algebras, faithfully represented on Hilbert spaces \mathcal{H} and \mathcal{H}' respectively, and let $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$ be a Hilbert space representation of \mathcal{A} . Then the kernel of Hilbert space representation*

$$\pi \otimes \iota_{\mathcal{B}} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B}(\mathcal{K} \otimes \mathcal{H}')$$

of the minimal tensor product $\mathcal{A} \otimes \mathcal{B}$, is $\ker \pi \otimes \mathcal{B}$.

Proof Consider a positive bounded approximate identity $\alpha \rightarrow u_{\alpha}$ of $\ker \pi$. This is a net strongly convergent to the orthogonal projection P on the closed subspace generated by $\ker \pi \mathcal{H}$. Since that subspace is left invariant by \mathcal{A} , P lies in the commutant of \mathcal{A} in $\mathcal{B}(\mathcal{H})$. Since, for all $a \in \mathcal{A}$,

$$\|\pi(a)\| = \lim_{\alpha} \|a - au_{\alpha}\| = \|a(I - P)\|,$$

the map

$$\pi(a) \in \pi(\mathcal{A}) \rightarrow a(I - P) \in \mathcal{B}(\mathcal{H})$$

is a faithful $*$ -representation of $\pi(\mathcal{A})$ on \mathcal{H} . The tensor product representation $\pi \otimes \iota_{\mathcal{B}}$ from the minimal tensor product $\mathcal{A} \otimes \mathcal{B}$ to operators on $\mathcal{K} \otimes \mathcal{H}'$, has range $*$ -isomorphic to the minimal tensor product $\pi(\mathcal{A}) \otimes \mathcal{B}$. Therefore, if $a_1, \dots, a_N \in \mathcal{A}$, $b_1, \dots, b_N \in \mathcal{B}$, the quotient norm of $[\sum_1^N a_i \otimes b_i]$ with respect

to the ideal $\ker(\pi \otimes \iota)$ coincides with the norm of the image $\sum_1^N \pi(a_i) \otimes b_i$ in the minimal tensor product $\pi(\mathcal{A}) \otimes \mathcal{B}$. In that C^* -algebra, by Turumaru's theorem (see, e.g. [8], Prop. 1.22.9), asserting that the tensor product of faithful representations is faithful, and therefore isometric,

$$\left\| \sum_1^N \pi(a_i) \otimes b_i \right\| = \left\| \sum_1^N a_i(I - P) \otimes b_i \right\|,$$

where at the right hand side we are using the operator norm arising from the Hilbert space $\mathcal{H} \otimes \mathcal{H}'$. Thus the right hand side equals

$$\lim_{\alpha} \left\| \sum_1^N a_i(I - u_{\alpha}) \otimes b_i \right\| = \lim_{\alpha} \left\| \sum_1^N a_i \otimes b_i - \left(\sum_1^N a_i \otimes b_i \right) (u_{\alpha} \otimes I) \right\|.$$

Since $\alpha \rightarrow u_{\alpha} \otimes I$ is an approximate identity for $\ker \pi \otimes \mathcal{B}$, the last term equals, in turn, the quotient norm of $[\sum_1^N a_i \otimes b_i]$ with respect to the ideal $\ker \pi \otimes \mathcal{B}$. Therefore we can conclude that the natural $*$ -homomorphism

$$\mathcal{A} \otimes \mathcal{B} / \ker \pi \otimes \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B} / \ker(\pi \otimes \iota_{\mathcal{B}})$$

is isometric.

2.4 Definition Let \mathcal{J} be a closed ideal of \mathcal{A} . We shall call \mathcal{J} a *closed coideal* if $\Delta(\mathcal{J}) \subseteq \mathcal{J} \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{J}$.

Recall that the sum of a closed ideal and a C^* -subalgebra in a C^* -algebra \mathcal{B} is always a C^* -subalgebra of \mathcal{B} (Cor. 1.5.8 in [5]). So the sum $\mathcal{J} \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{J}$ is a closed ideal of $\mathcal{A} \otimes \mathcal{A}$.

2.5 Proposition If $K = (\mathcal{A}', \Delta')$ is a compact quantum subgroup of $G = (\mathcal{A}, \Delta)$ defined by the surjection $\pi : \mathcal{A} \rightarrow \mathcal{A}'$, $\ker \pi$ is a closed ideal and a closed coideal of \mathcal{A} .

Proof For any $a \in \ker \pi$, the element $b := \iota_{\mathcal{A}} \otimes \pi(\Delta(a)) \in \mathcal{A} \otimes \mathcal{A}'$ lies in the kernel of $\pi \otimes \iota_{\mathcal{A}'}$, as

$$\pi \otimes \iota_{\mathcal{A}'}(b) = \pi \otimes \pi \Delta(a) = \Delta'(\pi(a)) = 0.$$

Since

$$\ker(\pi \otimes \iota_{\mathcal{A}'}) = \ker \pi \otimes \mathcal{A}' = \text{Image}(\iota_{\ker \pi} \otimes \pi),$$

there is $c \in \ker \pi \otimes \mathcal{A}$ such that

$$\iota_{\mathcal{A}} \otimes \pi(\Delta(a)) = (\iota_{\ker \pi} \otimes \pi)(c).$$

Set $d := \Delta(a) - c$. Since

$$\iota_{\mathcal{A}} \otimes \pi(d) = (\iota_{\mathcal{A}} \otimes \pi)(\Delta(a)) - (\iota_{\ker \pi} \otimes \pi)(c) = 0,$$

$$d \in \ker(\iota_{\mathcal{A}} \otimes \pi) = \mathcal{A} \otimes \ker \pi.$$

So

$$\Delta(a) \in \mathcal{A} \otimes \ker \pi + \ker \pi \otimes \mathcal{A}.$$

2.6 Theorem *If \mathcal{J} is a closed ideal and a closed coideal of \mathcal{A} , there is a unique coproduct $\Delta^{\mathcal{J}}$ on \mathcal{A}/\mathcal{J} making $K := (\mathcal{A}/\mathcal{J}, \Delta^{\mathcal{J}})$ into a compact quantum subgroup of G through the quotient map $q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$. Any compact quantum subgroup of G is of this form.*

Proof The the quotient C^* -algebra $\mathcal{A}' := \mathcal{A}/\mathcal{J}$ can be endowed with the unital $*$ -homomorphism

$$\widehat{\Delta} : \mathcal{A}' \rightarrow (\mathcal{A} \otimes \mathcal{A})/\mathcal{J},$$

with $\mathcal{J} := \mathcal{J} \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{J}$, taking the element $a + \mathcal{J}$ to the element $\Delta(a) + \mathcal{J}$. On the other hand if $q : \mathcal{A} \rightarrow \mathcal{A}'$ is the quotient map, then the $*$ -epimorphism

$$q \otimes q : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}' \otimes \mathcal{A}'$$

actually annihilates \mathcal{J} , and therefore it gives rise to a $*$ -epimorphism

$$\widehat{q \otimes q} : (\mathcal{A} \otimes \mathcal{A})/\mathcal{J} \rightarrow \mathcal{A}' \otimes \mathcal{A}'.$$

Set

$$\Delta^{\mathcal{J}} := \widehat{q \otimes q} \circ \widehat{\Delta} : \mathcal{A}' \rightarrow \mathcal{A}' \otimes \mathcal{A}'.$$

One has:

$$\Delta^{\mathcal{J}}(q(a)) = q \otimes q(\Delta(a)).$$

By the previous proposition, any compact quantum subgroup of G is of this form, with \mathcal{J} the kernel of the defining surjective map.

3 Equivalent subgroups

We have noticed (Remark 2) that the definition of quantum subgroup we have given in the previous section has the disadvantage that the compact quantum group G_h becomes a proper quantum subgroup of G in the case where the Haar measure h is not faithful. In this section we introduce the notion of equivalence between compact quantum subgroups which has the effect that G_h becomes equivalent to the whole group G .

Recall that a N -dimensional unitary representation of a compact quantum group $G = (\mathcal{A}, \Delta)$ is a unitary matrix $u = (u_{i,j}) \in \text{Mat}_N(\mathcal{A})$ such that $\Delta(u_{i,j}) = \sum_{r=1}^N u_{i,r} \otimes u_{r,j}$. Let H be an N -dimensional Hilbert space with orthonormal basis $\{\psi_i, i = 1, \dots, N\}$. Consider the right Hilbert \mathcal{A} -module $H \otimes \mathcal{A}$, with inner product $(\psi \otimes a, \phi \otimes b)_{\mathcal{A}} = (\psi, \phi) a^* b$, $a, b \in \mathcal{A}$, $\psi, \phi \in H$. The linear map $u : \psi_i \in H \rightarrow \sum_j \psi_j \otimes u_{j,i} \in H \otimes \mathcal{A}$ satisfies:

$$(u(\psi), u(\phi))_{\mathcal{A}} = (\psi, \phi)_{\mathbb{C}} I, \psi, \phi \in H, \quad (3.1)$$

$$u \otimes \iota \circ u = \iota \otimes \Delta \circ u, \quad (3.2)$$

$$u(H)\mathcal{A} = H \otimes \mathcal{A}. \quad (3.3)$$

Conversely, any linear map $u : H \rightarrow H \otimes \mathcal{A}$ satisfying (3.1)–(3.3) arises from a unitary representation of G .

If H is infinite dimensional, the most general notion of unitary representation has been given in [1]. More in detail, we consider the Hilbert space $L^2(\mathcal{A})$ obtained completing \mathcal{A} w.r.t. the inner product defined by the Haar measure. A unitary representation of G on H is a unitary operator W on the tensor product Hilbert space $H \otimes L^2(\mathcal{A})$ satisfying on $H \otimes L^2(\mathcal{A}) \otimes L^2(\mathcal{A})$ the relation

$$W_{12}W_{13}V_{23} = V_{23}W_{12}.$$

Here V denote the multiplicative unitary on $L^2(\mathcal{A}) \otimes L^2(\mathcal{A})$ defined by $V(a \otimes b) = \Delta(a)I \otimes b$. Notice that equations (3.1) and (3.2) make sense even in the case where H is separable and infinite dimensional. We illustrate how one can get a representation of V from them.

Let us consider the natural continuous map τ from the right \mathcal{A} -Hilbert module $H \otimes \mathcal{A}$ to the Hilbert space $H \otimes L^2(\mathcal{A})$. A map u satisfying (3.1) and (3.2) gives rise to a map W_u on $H \otimes L^2(\mathcal{A})$ defined by

$$W_u \psi \otimes a = \tau(u(\psi)a), \quad \psi \in H, a \in \mathcal{A} \subset L^2(\mathcal{A}).$$

We shall need the following fact in Sect. 5.

3.1 Proposition *Let H be a separable Hilbert space, and $u : H \rightarrow H \otimes \mathcal{A}$ a linear map satisfying (3.1) and (3.2). If the set $\{u(\psi)a, \psi \in H, a \in \mathcal{A}\}$ is total in $H \otimes \mathcal{A}$, the map W_u defined as above is a unitary representation of the multiplicative unitary V .*

3.2 Definition Let u be a unitary finite dimensional Hilbert space representation of a compact quantum group G on H , and let $K = (\mathcal{A}', \Delta')$ be a compact subgroup of G , defined by $\pi : \mathcal{A} \rightarrow \mathcal{A}'$. Then $u \upharpoonright_K := \iota \otimes \pi \circ u : H \rightarrow H \otimes \mathcal{A}'$ is easily seen to be a unitary representation of K on H , that we call the *restriction* of u to K .

The space of *intertwining operators* (u, u') between two unitary representations on Hilbert spaces H and H' respectively, is the set of all linear maps T from H to H' such that

$$u' \circ T = T \otimes \iota \circ u,$$

with ι the identity map on \mathcal{A} . The category $\text{Rep}(G)$ with objects finite dimensional unitary G -representations and arrows intertwining operators is known to be a tensor C^* -category with conjugates [10].

Let us consider the functor from $\text{Rep}(G)$ to $\text{Rep}(K)$, taking a representation u of G to the restricted representation $u \upharpoonright_K$, and acting trivially on the arrows. This is clearly a faithful tensor $*$ -functor. If we identify each representation with its Hilbert space, this functor gives us an inclusion

$$\text{Rep}(G) \subset \text{Rep}(K)$$

of Hilbert space categories.

3.3 Theorem *If K is a compact quantum subgroup of G , the smallest full tensor $*$ -subcategory of $\text{Rep}(K)$ with subobjects and direct sums containing all the restricted representations $\{u \downarrow_K, u \in \text{Rep}(G)\}$, is the whole $\text{Rep}(K)$.*

In order to prove this theorem we shall need the smooth part \mathcal{A}_∞ of \mathcal{A} , the set of all linear combinations of matrix elements of all finite dimensional unitary representations of G . This is a dense $*$ -subalgebra of \mathcal{A} such that $\Delta(\mathcal{A}_\infty) \subset \mathcal{A}_\infty \odot \mathcal{A}_\infty$ (see [12]). Here \odot denotes the algebraic tensor product.

3.4 Proposition *There are choices of complete sets of inequivalent, unitary, irreducible representations $\hat{G} = \{u^\alpha, \alpha \in A\}$ and $\hat{K} = \{v^\beta, \beta \in B\}$ of G and K respectively, such that for each $\alpha \in A$, $u^\alpha \downarrow_K$ splits into a direct sum of elements of \hat{K} : $u^\alpha \downarrow_K = m_1 v^{\beta_1} \oplus \cdots \oplus m_N v^{\beta_N}$, with $\beta_1, \dots, \beta_N \in B$ and m_1, \dots, m_N positive integers. Every element of \hat{K} arises in this way.*

Proof Let us choose a complete set $\hat{G} = \{u^\alpha, \alpha \in A\}$ of irreducible unitary representations of G . Since the representation coefficients $u_{i,j}^\alpha$ span \mathcal{A}_∞ , which is dense in \mathcal{A} , the set $\{\pi(u_{i,j}^\alpha), \alpha \in A, i, j = 1, \dots, \dim(u^\alpha)\}$ span a dense $*$ -subalgebra of \mathcal{A}' . On the other hand each $\pi(u_{i,j}^\alpha)$ is the coefficient of the restricted representation $u \downarrow_K$ of u to K . Let us consider a complete set $\hat{K} = \{v^\beta, \beta \in B\}$ of unitary irreducible representations of K . Up to replacing u^α by a unitarily equivalent G -representation, we can assume that there exist $\beta_1, \dots, \beta_N \in B$ and multiplicities $m_1, \dots, m_N \geq 1$ such that $u^\alpha \downarrow_K = m_1 v^{\beta_1} \oplus \cdots \oplus m_N v^{\beta_N}$. Since the set of all the coefficients of the v 's thus obtained is dense in \mathcal{A}' , when α ranges over A , we must obtain all the irreducibles of K .

Theorem 3.3 is now an easy consequence of the previous proposition.

3.5 Definition Let $G = (\mathcal{A}, \Delta)$ be a compact quantum group. Two compact quantum subgroups of G , $K_1 = (\mathcal{A}', \Delta')$ and $K_2 = (\mathcal{A}'', \Delta'')$, defined by surjections $\pi_1 : \mathcal{A} \rightarrow \mathcal{A}'$ and $\pi_2 : \mathcal{A} \rightarrow \mathcal{A}''$ respectively, will be called *equivalent* if their smooth parts \mathcal{A}'_∞ and \mathcal{A}''_∞ are isomorphic as Hopf $*$ -algebras.

3.6 Proposition *K_1 and K_2 are equivalent subgroups of G if and only if $\text{Rep}(K_1) = \text{Rep}(K_2)$ as Hilbert space categories.*

Proof It is easy to show that an isomorphism between the smooth parts of two subgroups induces an isomorphism of tensor $*$ -categories, between the corresponding representation categories leaving fixed the representation Hilbert spaces. Conversely, if $\text{Rep}(K_1) = \text{Rep}(K_2)$ as Hilbert space categories, the smooth parts of K_1 and K_2 must be isomorphic as Hopf $*$ -algebras, as each of them is isomorphic to the Tannaka–Krein dual of that category, by Woronowicz Tannaka–Krein duality theorem [11].

Consider the smooth part of the kernel of $\pi : \mathcal{A} \rightarrow \mathcal{A}'$, $\mathcal{I}_\infty := \ker \pi \cap \mathcal{A}_\infty$, which is clearly a $*$ -ideal of \mathcal{A}_∞ .

Remark 3 \mathcal{I}_∞ is not dense in $\ker \pi$ in general. Indeed, consider the case of the quantum subgroup G_h defined by the representation $\pi_h : \mathcal{A} \rightarrow \pi_h(\mathcal{A})$ of G . If

π_h is not faithful (or, equivalently, h is not faithful), $\mathcal{J}_\infty = \{0\}$ by [10], but $\ker \pi_h \neq \{0\}$.

3.7 Lemma \mathcal{J}_∞ is a $*$ -ideal of \mathcal{A}_∞ and an algebraic coideal of $(\mathcal{A}_\infty, \Delta)$, in the sense that $\Delta(\mathcal{J}_\infty) \subset \mathcal{J}_\infty \odot \mathcal{A}_\infty + \mathcal{A}_\infty \odot \mathcal{J}_\infty$.

Proof The only nontrivial statement is that $\Delta(\mathcal{J}_\infty) \subset \mathcal{J}_\infty \odot \mathcal{A}_\infty + \mathcal{A}_\infty \odot \mathcal{J}_\infty$. We choose \hat{G} and \hat{K} as in the previous theorem. Let us call “diagonal” all the pairs (i, j) corresponding to the entries in the matrix u^α corresponding to the entries occupied by the coefficients of the v^{β_p} ’s in the matrix $u^\alpha \upharpoonright_K$, and “off diagonal” the remaining pairs. Let now $X \in \mathcal{J}_\infty$ be written uniquely as a linear combination of the coefficients of a finite set $\{u^\alpha, \alpha \in F\}$ of G -representations, i.e.

$$\begin{aligned} X &= \sum_{\alpha \in F} \sum_{i,j=1}^{\dim u^\alpha} \lambda_{i,j}^\alpha u_{i,j}^\alpha = \\ &= \sum_{\alpha \in F} \sum_{\text{diagonal pairs}} \lambda_{i,j}^\alpha u_{i,j}^\alpha + \sum_{\alpha \in F} \sum_{\text{off diagonal pairs}} \lambda_{i,j}^\alpha u_{i,j}^\alpha = \\ &= X' + X''. \end{aligned}$$

Now, if (i, j) is off diagonal, $\pi(u_{i,j}^\alpha) = 0$, therefore $u_{i,j}^\alpha \in \ker \pi$, which shows that X'' , and therefore also X' , lie in $\ker \pi$ as well. Therefore we are left to show that both $\Delta(u_{i,j}^\alpha)$, for all (i, j) off diagonal, and $\Delta(X')$ belong to $\mathcal{J}_\infty \odot \mathcal{A}_\infty + \mathcal{A}_\infty \odot \mathcal{J}_\infty$. Since, for (i, j) off diagonal, we can write

$$\Delta(u_{i,j}^\alpha) = \sum_{k:(i,k) \text{ is off diagonal}} u_{i,k}^\alpha \otimes u_{k,j}^\alpha + \sum_{k:(i,k) \text{ is diagonal}} u_{i,k}^\alpha \otimes u_{k,j}^\alpha,$$

we realize that $\Delta(u_{i,j}^\alpha) \in \mathcal{J}_\infty \odot \mathcal{A}_\infty + \mathcal{A}_\infty \odot \mathcal{J}_\infty$ (notice that in the second sum each (k, j) must be off diagonal, as (i, j) is).

Let us now think of the element $X' \in \ker \pi$. Let $F' = \{\beta_1, \dots, \beta_N\}$ denote the finite subset of B of all unitary irreducible K -representations obtained from the irreducible components of every representation in the set $\{u^\alpha \upharpoonright_K, \alpha \in F\}$. For each $r = 1, \dots, N$ consider the subset $F_r \subset F$ constituted by all $\alpha \in F$ for which v^{β_r} is a subrepresentation of $u^\alpha \upharpoonright_K$. For each such α , let m_r^α be the multiplicity of $v_r^{\beta_r}$ in $u^\alpha \upharpoonright_K$. We choose unitary equivalences so that for each $\alpha \in F_r$, the subrepresentations of the form $m_j^\alpha v^{\beta_j}$ of $u^\alpha \upharpoonright_K$ are listed in order with increasing indices j . Then

$$\begin{aligned} 0 = \pi(X') &= \sum_{\alpha \in F} \sum_{(i,j) \text{ diagonal}} \lambda_{i,j}^\alpha \pi(u_{i,j}^\alpha) = \\ &= \sum_{r=1}^N \sum_{h,k=1}^{\dim v_r^\beta} \left(\sum_{\alpha \in F_r} \sum_{p=1}^{m_r^\alpha} \lambda_{i_p^\alpha, j_p^\alpha}^\alpha \right) v_{h,k}^{\beta_r}, \end{aligned}$$

where, denoting by f_j the dimension of v^{β_j} , and with $\delta_r^\alpha := \sum_{j=1}^{r-1} m_j^\alpha f_j$, we have set: $i_1^\alpha = \delta_r^\alpha + h$, $i_2^\alpha = \delta_r^\alpha + f_r + h$, \dots , $i_{m_r^\alpha}^\alpha = \delta_r^\alpha + (m_r^\alpha - 1)f_r + h$,

$j_1^\alpha = \delta_r^\alpha + k$, $j_2^\alpha = \delta_r^\alpha + f_r + k$, \dots , $j_{m_r^\alpha}^\alpha = \delta_r^\alpha + (m_r^\alpha - 1)f_r + k$. Since the $v_{h,s}^\beta$'s are linearly independent, we see that for each possible r , and each possible pair $(h, k) \in \{1, \dots, f_r\} \times \{1, \dots, f_r\}$,

$$\sum_{\alpha \in F_r} \sum_{p=1}^{m_r^\alpha} \lambda_{i_p, j_p}^\alpha = 0.$$

Then

$$X' = \sum_r \sum_{h,k} X_{h,k}^r,$$

where

$$X_{h,k}^r = \sum_{\alpha \in F_r} \sum_{p=1}^{m_r^\alpha} \lambda_{i_p, j_p}^\alpha u_{i_p, j_p}^\alpha,$$

all of them belonging to $\ker \pi$. Therefore it suffices to show that

$$\Delta(X_{h,k}^r) \in \mathcal{I}_\infty \odot \mathcal{A}_\infty + \mathcal{A}_\infty \odot \mathcal{I}_\infty$$

for each fixed r, h, k . Set, for simplicity, $\mu_p^\alpha := \lambda_{i_p, j_p}^\alpha$. Then

$$\Delta(X_{h,k}^r) = \sum_{\alpha \in F_r} \sum_{p=1}^{m_r^\alpha} \sum_{q=1}^{\dim u^\alpha} \mu_p^\alpha u_{i_p, q}^\alpha \otimes u_{q, j_p}^\alpha.$$

We split the sum in q in two parts: the sum with q ranging the interval

$$\delta_r^\alpha + (p-1)f_r + 1, \dots, \delta_r^\alpha + (p-1)f_r + f_r = \delta_r^\alpha + pf_r$$

and the sum over the remaining values. We get

$$\begin{aligned} & \sum_{\alpha \in F_r} \sum_{p=1}^{m_r^\alpha} \sum_{q=\delta_r^\alpha + (p-1)f_r + 1}^{\delta_r^\alpha + (p-1)f_r + p} \mu_p^\alpha u_{i_p, q}^\alpha \otimes u_{q, j_p}^\alpha + \\ & \sum_{\alpha \in F_r} \sum_{p=1}^{m_r^\alpha} \sum_{\text{the remaining indices } q} \mu_p^\alpha u_{i_p, q}^\alpha \otimes u_{q, j_p}^\alpha. \end{aligned}$$

Each pair (i_p, q) and (q, j_p) arising from the second sum is off diagonal, so the second sum belongs to $\mathcal{I}_\infty \odot \mathcal{I}_\infty$. Let us think of the first sum. We first perform the sum over the pairs (i_p, q) in the set

$$A_1 := \{(i_1, \delta_r^\alpha + 1), (i_2, \delta_r^\alpha + f_r + 1), \dots, (i_{m_r^\alpha}, \delta_r^\alpha + (m_r^\alpha - 1)f_r + 1)\},$$

followed by the sum over the pairs

$$A_2 := \{(i_1, \delta_r^\alpha + 2), (i_2, \delta_r^\alpha + f_r + 2), \dots, (i_{m_r^\alpha}, \delta_r^\alpha + (m_r^\alpha - 1)f_r + 2)\},$$

and, finally at the last step, over the pairs in the set

$$A_{f_r} := \{(i_1, \delta_r^\alpha + f_r), (i_2, \delta_r^\alpha + f_r + f_r), \dots, (i_{m_r}^\alpha, \delta_r^\alpha + (m_r^\alpha - 1)f_r + f_r)\}.$$

We show then that each addendum

$$\sum_{\alpha \in F_r} \sum_{(i_p, q) \in A_s} \mu_p^\alpha u_{i_p, q}^\alpha \otimes u_{q, j_p}^\alpha$$

lies in $\mathcal{J}_\infty \odot \mathcal{A}_\infty + \mathcal{A}_\infty \odot \mathcal{J}_\infty$. In fact, assuming for simplicity $s = 1$, that sum can be also written as

$$\left(\sum_{\alpha \in F_r} \sum_{(i_p, q) \in A_1} \mu_p^\alpha u_{i_p, q}^\alpha \right) \otimes u_{\delta_r^\alpha + 1, j_1}^\alpha + \sum_{\alpha \in F_r} \sum_{(i_p, q) \in A_1} \mu_p^\alpha u_{i_p, q}^\alpha \otimes (u_{q, j_p}^\alpha - u_{\delta_r^\alpha + 1, j_1}^\alpha),$$

and the claim is proved.

Remark 4 Let us consider complete sets \hat{G} , \hat{K} of irreducible representations of G and K respectively, as in proposition 3.4. We thus see that π restricts to a surjective $*$ -homomorphism $\mathcal{A}_\infty \rightarrow \mathcal{A}'_\infty$ with kernel \mathcal{J}_∞ . Since \mathcal{J}_∞ is an algebraic ideal and coideal of the Hopf $*$ -algebra $(\mathcal{A}_\infty, \Delta)$, $(\mathcal{A}'_\infty, \Delta')$ and $(\mathcal{A}_\infty/\mathcal{J}_\infty, \Delta^{\mathcal{J}_\infty})$ are isomorphic as Hopf $*$ -algebras.

Remark 5 We show that up to replacing K with an equivalent subgroup, K_{\max} , we can always assume that the smooth part of the \mathcal{J}_∞ be dense in $\ker \pi$. If $K = (\mathcal{A}', \Delta')$ is a compact quantum subgroup of $G = (\mathcal{A}, \Delta)$, we can complete the $*$ -ideal $\mathcal{J}_\infty = \ker \pi \cap \mathcal{A}_\infty$ and obtain a closed ideal and a closed coideal \mathcal{J}_∞^- . Then thanks to Theorem 2.6 and Lemma 3.7, we can form another compact quantum subgroup K_{\max} by taking the quotient with respect to \mathcal{J}_∞^- . Let q be the corresponding quotient map. This subgroup is clearly equivalent to the original subgroup K . But K can be in turn regarded as a subgroup of K_{\max} , as the inclusion $\mathcal{J}_\infty^- \subset \ker \pi$ provides a $*$ -epimorphism

$$\alpha : \mathcal{A}/\mathcal{J}_\infty^- \rightarrow \mathcal{A}/\ker \pi = \mathcal{A}'$$

such that, for $a \in \mathcal{A}$,

$$\Delta' \circ \alpha(a + \mathcal{J}_\infty^-) = \Delta'(\pi(a)) = \pi \otimes \pi \circ \Delta(a).$$

On the other hand

$$\alpha \otimes \alpha \circ \Delta^{\mathcal{J}_\infty^-}(a + \mathcal{J}_\infty^-) = \alpha \circ q \otimes \alpha \circ q \Delta(a) = \pi \otimes \pi \circ \Delta(a),$$

so $\Delta' \circ \alpha = \alpha \otimes \alpha \circ \Delta^{\mathcal{J}_\infty^-}$.

Let h' be the Haar measure on K . The associated GNS representation $\pi_{h'} : \mathcal{A}' \rightarrow \pi_{h'}(\mathcal{A}')$ has as image the compact quantum group $K_{h'}$, which is a subgroup of K . Notice that α is an isomorphism on the smooth part of $\mathcal{A}/\mathcal{J}_\infty^-$. Therefore composing $h' \circ \alpha$ gives a positive state on $\mathcal{A}/\mathcal{J}_\infty^-$ which acts as the Haar measure h'_{\max} of K_{\max} . Thus $L^2(K_{\max}, h'_{\max}) = L^2(K, h')$ and $\pi_{h'} \circ \alpha = \pi_{h'_{\max}}$.

These arguments show that if h'_{\max} is faithful then both $\pi_{h'}$ and α are faithful maps, so the equivalence class of subgroups of G equivalent to K is constituted by all $*$ -isomorphic quantum groups.

We have seen that if K is a compact quantum subgroup of G then $\text{Rep}(G) \subset \text{Rep}(K)$ as a tensor C^* -category and that equivalent subgroups have the same representation categories (Theorem 3.7). Therefore there is a well defined map associating to the equivalence class $[K]$ of a subgroup K of G a Hilbert space category, $\text{Rep}(K)$, containing $\text{Rep}(G)$.

3.7 Theorem *Let G be a compact quantum group. The map $[K] \rightarrow \text{Rep}(K)$ is a bijective correspondence between equivalence classes of quantum subgroups of G and Hilbert space $*$ -categories with tensor products, conjugates, direct sums and subobjects, containing $\text{Rep}(G)$.*

Proof We just need to show that the map is surjective. Let \mathcal{T} be a tensor $*$ -category containing $\text{Rep}(G)$ as in the statement. Then by Woronowicz Tannaka-Krein duality theorem we can find a unital Hopf $*$ -algebra $K_0 = (\mathcal{B}, \Delta')$ with a unital coassociative coproduct Δ' which is a universal model for \mathcal{T} (see [11]). On the other hand, since $\mathcal{T} \supset \text{Rep}(G)$, by universality of $(\mathcal{A}_\infty, \Delta)$ there must exist a surjective $*$ -homomorphism $\mathcal{A}_\infty \rightarrow \mathcal{B}$ intertwining the corresponding coproducts. The kernel \mathcal{I}_0 is a $*$ -ideal and also a coideal (use the same arguments as in the proof of Lemma 3.7 to show that it is a coideal). Complete \mathcal{I}_0 in \mathcal{A} and obtain a closed ideal and a closed coideal \mathcal{I} in \mathcal{A} . The corresponding quantum subgroup K of G has then \mathcal{T} as representation category.

4 Quantum quotient spaces

Recall that an *action* of a compact quantum group $G = (\mathcal{A}, \Delta)$ on a unital C^* -algebra \mathcal{F} is a unital $*$ -homomorphism $\delta : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{A}$ such that

$$\delta \otimes \iota \circ \delta = \iota \otimes \Delta \circ \delta.$$

The map δ will be called *ergodic* if the fixed point algebra

$$\mathcal{F}^\delta := \{f \in \mathcal{F} : \delta(f) = f \otimes I\}$$

reduces to the complex numbers.

We call a unital $*$ -homomorphism $\delta' : \mathcal{F} \rightarrow \mathcal{A} \otimes \mathcal{F}$ an *opposite action* of G if it satisfies the relation

$$\iota \otimes \delta' \circ \delta' = \Delta \otimes \iota \circ \delta'.$$

δ' is an opposite action of G on \mathcal{F} if and only if $\delta := \vartheta(\mathcal{F}, \mathcal{A}) \circ \delta'$ is an action of the opposite group G_o on \mathcal{F} .

4.1 Proposition *If $K = (\mathcal{A}', \Delta')$ is a compact quantum subgroup of $G = (\mathcal{A}, \Delta)$, there is an action*

$$\delta_K : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}'$$

of K on the C^* -algebra \mathcal{A} , defined by $\delta_K := \iota \otimes \pi \circ \Delta$.

Proof Indeed,

$$\begin{aligned} \delta_K \otimes \iota \circ \delta_K &= \iota \otimes \pi \otimes \iota \circ \Delta \otimes \iota \circ \iota \otimes \pi \circ \Delta = \\ \iota \otimes \pi \otimes \pi \circ \Delta \otimes \iota \circ \Delta &= \iota \otimes \pi \otimes \pi \circ \iota \otimes \Delta \circ \Delta = \\ \iota \otimes \Delta' \circ \iota \otimes \pi \circ \Delta &= \iota \otimes \Delta' \circ \delta_K. \end{aligned}$$

We then consider the fixed point algebra

$$\mathcal{A}^{\delta_K} := \{T \in \mathcal{A} : \delta_K(T) = T \otimes I\}$$

and call it *the quantum left coset space*. Similarly, $\delta'_K := \pi \otimes \iota \circ \Delta$ is an opposite action of K on \mathcal{A} , whose fixed point algebra

$$\mathcal{A}^{\delta'_K} := \{T \in \mathcal{A} : \delta'_K(T) = I \otimes T\}$$

will be called *the quantum right coset space*.

Let G be a compact group, and $(\mathcal{C}(G), \Delta)$ the associated compact quantum group with coproduct $\Delta(f)(g, h) = f(gh)$. Let K be a closed subgroup of G . The action of the quantum subgroup group $(\mathcal{C}(K), \Delta')$ on $\mathcal{C}(G)$ just defined is given by

$$\delta_K(f)(g, k) = f(gk), \quad g \in G, \quad k \in K.$$

Thus the fixed point algebra is

$$\mathcal{C}(G)^{\delta_K} = \{f \in \mathcal{C}(G) : f(gk) = f(g), g \in G, k \in K\},$$

the set of all class functions $\mathcal{C}(G/K)$. Its spectrum is the compact space G/K of left cosets. Similarly, $\delta'_K(f)(k, g) = f(kg)$, so

$$\mathcal{C}(G)^{\delta'_K} = \{f \in \mathcal{C}(G) : f(kg) = f(g), g \in G, k \in K\} = \mathcal{C}(K \backslash G)$$

with spectrum the space $K \backslash G$ of right cosets.

These coset spaces are known to be endowed with left and right G -actions by homeomorphisms. For example, for the right coset space,

$$g : Kg_1 \in K \backslash G \rightarrow Kg_1g \in K \backslash G$$

makes $K \backslash G$ into a homogeneous space, in the sense that the above G -action is ergodic. There is a natural way of defining the corresponding G -actions in the quantum situation: restrict the coproduct Δ to the left and right coset spaces.

4.2 Proposition [9]

a) *The map*

$$\eta := \Delta \upharpoonright_{\mathcal{A}^{\delta_K}} : \mathcal{A}^{\delta_K} \rightarrow \mathcal{A} \otimes \mathcal{A}^{\delta_K}$$

is an ergodic opposite action of G on the quantum left coset space \mathcal{A}^{δ_K} such that $\mathcal{A} \otimes I\eta(\mathcal{A}^{\delta_K})$ is dense in $\mathcal{A} \otimes \mathcal{A}^{\delta_K}$.

b) *The map*

$$\eta' := \Delta \upharpoonright_{\mathcal{A}^{\delta'_K}} : \mathcal{A}^{\delta'_K} \rightarrow \mathcal{A}^{\delta'_K} \otimes \mathcal{A}$$

is an ergodic action of G on the quantum right coset space $\mathcal{A}^{\delta'_K}$ such that $I \otimes \mathcal{A}\eta'(\mathcal{A}^{\delta'_K})$ is dense in $\mathcal{A}^{\delta'_K} \otimes \mathcal{A}$.

Proof We prove only a). Consider the map $\iota \otimes \delta_K : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}'$. This is an action of K on $\mathcal{A} \otimes \mathcal{A}$ with fixed point algebra $\mathcal{A} \otimes \mathcal{A}^{\delta_K}$. We show that $\eta(\mathcal{A}^{\delta_K}) \subset \mathcal{A} \otimes \mathcal{A}^{\delta_K}$, or, equivalently, that, for $T \in \mathcal{A}^{\delta_K}$, $\iota \otimes \delta_K(\Delta(T)) = \Delta(T) \otimes I$. Indeed,

$$\begin{aligned} \iota \otimes \delta_K(\Delta(T)) &= \iota \otimes \iota \otimes \pi \circ \iota \otimes \Delta \circ \Delta(T) = \\ \iota \otimes \iota \otimes \pi \circ \Delta \otimes \iota \circ \Delta(T) &= \Delta \otimes \iota \circ \delta_K(T) = \\ \Delta \otimes \iota(T \otimes I) &= \Delta(T) \otimes I. \end{aligned}$$

We show that the η -fixed point algebra reduces to the complex numbers. Let $T \in \mathcal{A}^{\delta_K}$ satisfy $\eta(T) = I \otimes T$. Since $\delta_K(T) = T \otimes I$, we have both:

$$\Delta(T) = I \otimes T, \iota \otimes \pi \circ \Delta(T) = T \otimes I,$$

so

$$I \otimes \pi(T) = T \otimes I,$$

which shows that $T \in \mathbb{C}$. We finally show that the linear span of elements of the form $a \otimes I\eta(b')$, with $a \in \mathcal{A}$, $b' \in \mathcal{A}^{\delta_K}$, is dense. Let h' be the Haar measure of K , and $E : \mathcal{A} \rightarrow \mathcal{A}^{\delta_K}$ the conditional expectation: $E(b) := \iota_{\mathcal{A}} \otimes h' \circ \delta_K$. A straightforward computation shows that E commutes with the action η , in the sense that

$$\eta \circ E = \iota_{\mathcal{A}} \otimes E \circ \Delta.$$

Take $b' \in \mathcal{A}^{\delta_K}$ of the form $E(b)$, with b ranging over \mathcal{A} . Then

$$a \otimes I\eta(b') = \iota_{\mathcal{A}} \otimes E(a \otimes I\Delta(b)),$$

and the conclusion follows from the fact that $\mathcal{A} \otimes I\Delta(\mathcal{A})$ is dense in $\mathcal{A} \otimes \mathcal{A}$.

The pairs

$$G/K := (\mathcal{A}^{\delta_K}, \eta)$$

and

$$K \backslash G := (\mathcal{A}^{\delta'_K}, \eta')$$

will be called *the compact quantum left and right quotient spaces*, respectively, defined by the subgroup K of G .

5 Induced representations and Frobenius reciprocity

In this section we define the representation of a compact quantum group induced by a representation of a compact quantum subgroup.

Let $K = (\mathcal{A}', \Delta')$ be a compact subgroup of G , defined by $\pi : \mathcal{A} \rightarrow \mathcal{A}'$. and let $u : H \rightarrow H \otimes \mathcal{A}'$ be a finite dimensional unitary representation of K on the Hilbert space H . Consider the opposite action of K on \mathcal{A} :

$$\delta'_K := \pi \otimes \iota \circ \Delta : \mathcal{A} \rightarrow \mathcal{A}' \otimes \mathcal{A}.$$

One has: $\delta'_K(\mathcal{A}_\infty) \subset \pi(\mathcal{A}_\infty) \odot \mathcal{A}_\infty \subset \mathcal{A}'_\infty \odot \mathcal{A}_\infty$. Define maps:

$$u \otimes \iota : H \otimes \mathcal{A} \rightarrow H \otimes \mathcal{A}' \otimes \mathcal{A},$$

$$\iota \otimes \delta'_K : H \otimes \mathcal{A} \rightarrow H \otimes \mathcal{A}' \otimes \mathcal{A}.$$

Note that $u \otimes \iota(H \odot \mathcal{A}_\infty) \subset H \odot \mathcal{A}'_\infty \odot \mathcal{A}_\infty$ and $\iota \otimes \delta'_K(H \odot \mathcal{A}_\infty) \subset H \odot \mathcal{A}'_\infty \odot \mathcal{A}_\infty$. Consider the following subspaces:

$$X_{\text{Ind}_\infty(u)} := \{\xi \in H \odot \mathcal{A}_\infty : u \otimes \iota(\xi) = \iota \otimes \delta'_K(\xi)\},$$

$$X_{\text{Ind}_0(u)} := \{\xi \in H \otimes \mathcal{A} : u \otimes \iota(\xi) = \iota \otimes \delta'_K(\xi)\},$$

the latter being a Banach subspace of $H \otimes \mathcal{A}$. Notice that if $\xi, \eta \in X_{\text{Ind}_0(u)}$, $(\xi, \eta)_\mathcal{A}$ is an element of the right coset space $\mathcal{A}^{\delta'_K}$:

$$\begin{aligned} \delta'_K((\xi, \eta)_\mathcal{A}) &= (\iota_H \otimes \delta'_K(\xi), \iota_H \otimes \delta'_K(\eta))_{\mathcal{A}' \otimes \mathcal{A}} = \\ &= (u \otimes \iota_\mathcal{A}(\xi), u \otimes \iota_\mathcal{A}(\eta))_{\mathcal{A}' \otimes \mathcal{A}} = I \otimes (\xi, \eta)_\mathcal{A} \end{aligned}$$

by (3.1). Therefore $X_{\text{Ind}_0(u)}$ is a right Hilbert $\mathcal{A}^{\delta'_K}$ -module.

We shall also consider the norm closure $H_{\text{Ind}(u)}$ in $H \otimes L^2(\mathcal{A})$ of the image of $X_{\text{Ind}_\infty(u)}$ under the natural continuous map $\tau : H \otimes \mathcal{A} \rightarrow H \otimes L^2(\mathcal{A})$.

One should point out that there do exist functions in $H_{\text{Ind}(u)}$. Indeed, let

$$E : H \odot \mathcal{A}_\infty \rightarrow H \odot \mathcal{A}_\infty$$

denote the linear map that takes a simple tensor $x = \psi \otimes a$ to the element

$$\begin{aligned} E(x) &:= \iota_H \otimes (h' \circ m) \otimes \iota_\mathcal{A}((\iota_H \otimes \kappa' \circ u) \otimes \delta'_K(x)) = \\ &= \iota_H \otimes (h' \circ m) \otimes \iota_\mathcal{A}((\iota_H \otimes \kappa' \circ u)(\psi) \otimes \delta'_K(a)), \end{aligned}$$

where $m : \mathcal{A}'_\infty \odot \mathcal{A}'_\infty \rightarrow \mathcal{A}'_\infty$ denotes the multiplication map: $a \otimes b \rightarrow ab$, h' the Haar measure for K and κ' the coinverse for \mathcal{A}'_∞ .

5.1 Lemma *E is an idempotent map with $E(H \odot \mathcal{A}_\infty) = X_{\text{Ind}_\infty(u)}$.*

Proof We show that, for any $x \in X_{\text{Ind}_\infty(u)}$, $E(x) = x$.

$$E(x) = \iota_H \otimes (h' \circ m) \otimes \iota_\mathcal{A}((\iota_H \otimes \kappa' \circ u) \otimes \iota_{\mathcal{A}'} \otimes \iota_\mathcal{A} \circ u \otimes \iota_\mathcal{A}(x)) =$$

$$\begin{aligned} \iota_H \otimes (h' \circ m) \otimes \iota_{\mathcal{A}}(\iota_H \otimes \kappa' \otimes \iota_{\mathcal{A}'} \otimes \iota_{\mathcal{A}} \circ \iota_H \otimes \Delta' \otimes \iota_{\mathcal{A}} \circ u \otimes \iota_{\mathcal{A}}(x)) = \\ \iota_H \otimes e' \otimes \iota_{\mathcal{A}} \circ u \otimes \iota_{\mathcal{A}}(x) = x. \end{aligned}$$

since $m \circ \kappa' \otimes \iota_{\mathcal{A}'} \circ \Delta' = e'$, with e' the counit of \mathcal{A}'_∞ , and $\iota_H \otimes e' \circ u = \iota_H$. We are left to show that for any $x \in H \odot \mathcal{A}_\infty$, $E(x) \in X_{\text{Ind}_\infty}(u)$.

$$\begin{aligned} \iota_H \otimes \delta'_K(E(x)) &= \iota_H \otimes (h' \circ m) \otimes \iota_{\mathcal{A}'} \otimes \iota_{\mathcal{A}} \circ \iota_H \otimes \iota_{\mathcal{A}'} \otimes \delta'_K((\iota_H \otimes \kappa' \circ u) \otimes \delta'_K(x)) = \\ &= \iota_H \otimes (h' \circ m) \otimes \iota_{\mathcal{A}'} \otimes \iota_{\mathcal{A}}((\iota_H \otimes \kappa' \circ u) \otimes (\iota_{\mathcal{A}'} \otimes \delta'_K \circ \delta'_K)(x)) = \\ &= \iota_H \otimes (h' \circ m) \otimes \iota_{\mathcal{A}'} \otimes \iota_{\mathcal{A}}((\iota_H \otimes \kappa' \circ u) \otimes (\Delta' \otimes \iota_{\mathcal{A}} \circ \delta'_K)(x)). \end{aligned}$$

Now for $x = \psi_i \otimes a$ with $a \in \mathcal{A}_\infty$ and ψ_i an element of an orthonormal basis of H , write $\delta'_K(a) = \sum a'_1 \otimes a_2$ and $u(\psi_i) = \sum_r \psi_r \otimes u_{r,i}$. Inserting these computations in the last term gives, using strong right invariance of the Haar measure [10]: $\kappa'(h' \otimes \iota(a \otimes I\Delta'(b))) = h' \otimes \iota(\Delta'(a)b \otimes I)$,

$$\begin{aligned} \sum_r \psi_r \otimes (h' \otimes \iota_{\mathcal{A}'}(\kappa'(u_{r,i}) \otimes I\Delta'(a'_1))) \otimes a_2 = \\ \sum_r \psi_r \otimes (\kappa'^{-1}(h' \otimes \iota_{\mathcal{A}'}(\Delta' \circ \kappa'(u_{r,i})a'_1 \otimes I))) \otimes a_2 = \\ \sum_r \psi_r \otimes (\kappa'^{-1}(\iota \otimes h'(\kappa' \otimes \kappa' \circ \Delta'(u_{r,i})I \otimes a'_1))) \otimes a_2 = \\ \sum_r \psi_r \otimes (\iota \otimes h'(\iota \otimes \kappa' \circ \Delta'(u_{r,i})I \otimes a'_1)) \otimes a_2 = \\ \sum_{r,k} \psi_r \otimes u_{r,k} \otimes h'(\kappa'(u_{k,i})a'_1)a_2 = \\ \sum_k u(\psi_k) \otimes h'(\kappa'(u_{k,i})a'_1)a_2 = u \otimes \iota_{\mathcal{A}}(E(x)), \end{aligned}$$

since $\Delta' \circ \kappa' = \vartheta \circ \kappa' \otimes \kappa' \circ \Delta'$ with ϑ the flip automorphism of $\mathcal{A}' \otimes \mathcal{A}'$.

5.2 Lemma Let $(f_z)_{z \in \mathbb{C}}$ and $(f'_z)_{z \in \mathbb{Z}}$ be the family of linear multiplicative functionals defined on \mathcal{A}_∞ and \mathcal{A}'_∞ respectively [10], and let u and v be irreducible unitary representations of K and G respectively. If $v \upharpoonright_K = \oplus_1^m u \oplus u'$ with u' disjoint from u , then for all $r, s = 1, \dots, N_u$, with N_u the dimension of u , and $j = 0, \dots, m-1$, there exist positive constants λ_j such that

$$f_{-1}(v_{r+jN_u, s+jN_u}) = \lambda_j f'_{-1}(u_{r,s}).$$

Proof Let F_v be the unique positive intertwiner from v to the double contra-gradient representation v^{cc} with $\text{Tr}(F_v) = \text{Tr}(F_v^{-1})$. It is easy to check that F_v is an intertwiner from $v \upharpoonright_K$ to $v \upharpoonright_K^{cc}$. Therefore F_v leaves globally invariant the space of $\oplus_1^m u$ and $F_j = E_j F_v \upharpoonright_{H_j}$ are positive invertible intertwiners from u to u^{cc} , with E_j the orthogonal projection from H_v to the j -th space H_j of u .

Therefore for some positive constants λ_j , $\lambda_j F_j = F_u$. By definition of f_{-1} (see (5.22) in [10]), following Woronowicz notation,

$$\begin{aligned} f_{-1}(v_{r+jN_u, s+jN_u}) &= \text{Tr}(F_v^{-1} m_{r+jN_u, s+jN_u}^v) = \\ \text{Tr}(F_j^{-1} m_{r,s}^u) &= \lambda_j \text{Tr}(F_u^{-1} m_{r,s}^u) = \lambda_j f'_{-1}(u_{r,s}). \end{aligned}$$

We next show that E is a selfadjoint projection.

5.3 Proposition *Let us regard $H \odot \mathcal{A}_\infty$ as a dense subspace of the Hilbert space $H \otimes L^2(\mathcal{A})$. Then the densely defined operator $E : H \odot \mathcal{A}_\infty \rightarrow H \otimes L^2(\mathcal{A})$ is Hermitian:*

$$(x', E(x)) = (E(x'), x), \quad x, x' \in H \odot \mathcal{A}_\infty. \quad (5.1)$$

Therefore E is bounded and extends uniquely to the orthogonal projection of $H \otimes L^2(\mathcal{A})$ onto $H_{\text{Ind}(u)}$.

Proof \mathcal{A}_∞ is linearly spanned by coefficients of unitary irreducible representations of G [10], therefore it suffices to take $x = \psi_i \otimes v_{p,q}$ and $x' = \psi_{i'} \otimes v'_{p',q'}$, with v and v' irreducible representations of G and $\{\psi_r\}$ an orthonormal basis of H . A straightforward computation shows that for $x = \psi_i \otimes v_{p,q}$,

$$E(x) = \sum_{r,k} \psi_r \otimes h'(\kappa'(u_{r,i})\pi(v_{p,k}))v_{k,q} = \sum_{r,k} h'(u_{i,r}^* \pi(v_{p,k}))\psi_r \otimes v_{k,q}. \quad (5.2)$$

Since any unitary representation of K is the direct sum of irreducibles and since E leaves globally invariant any subspace of the form $H' \odot \mathcal{A}_\infty$, with H' the space of a subrepresentation of u , it suffices to assume u irreducible. By the Peter-Weyl theory for compact quantum groups [10], $h'(u_{i,r}^* \pi(v_{p,k})) = 0$ unless $v \upharpoonright_K$ contains u as a subrepresentation. In that case the computation of $E(x)$ shows that $(x', E(x)) = 0$ unless $v' \upharpoonright_K$ contains u , again by the Peter-Weyl theory of K . Therefore both sides of (5.1) annihilate, and therefore coincide, unless both v and v' contain u as a subrepresentation when restricted to K . Let us assume then that this is the case. The computation of $E(x)$ shows that, up to replacing v by an equivalent representation, we can assume that $v \upharpoonright_K$ takes the form: $v \upharpoonright_K = \oplus_1^m u \oplus u'$, with m the multiplicity of u in $v \upharpoonright_K$. If p is bigger than mN_u , with N_u the dimension of u , $E(x) = 0$ since u' and u are disjoint. For $p = jN_u + 1, \dots, (j+1)N_u$, for some $j = 0, \dots, m-1$, $\pi(v_{p,k}) = 0$ for all k , unless $k = jN_u + 1, \dots, (j+1)N_u$. In this case $\pi(v_{p,k}) = u_{p-jN_u, k-jN_u}$. Then by [10], Theorem 5.7,

$$h(u_{i,r}^* \pi(v_{p,k})) = h(u_{i,r}^* u_{p-jN_u, k-jN_u}) = \delta_{r, k-jN_u} \frac{f'_{-1}(u_{p-jN_u, i})}{f'_1(\chi_u)},$$

with $z \rightarrow f_z$ the linear multiplicative functionals of \mathcal{A}'_∞ defined in [10], Theorem 5.6. Therefore

$$E(x) = \frac{f'_{-1}(u_{p-jN_u, i})}{f'_1(\chi_u)} \sum_r \psi_r \otimes v_{r+jN_u, q}.$$

Here $\chi_u = \sum_s u_{s,s}$ is the character of the representation u . Thus $E(x) = 0$ unless $q = jN_u + 1, \dots, (j+1)N_u$. Assume then that this is the case. Now

$$\begin{aligned} (x', E(x)) &= \frac{f'_{-1}(u_{p-jN_u, i})}{f'_1(\chi_u)} \sum_r \delta_{i', r} h(v'_{p', q'} * v_{r+jN_u, q}) = \\ &= \frac{f'_{-1}(u_{p-jN_u, i})}{f'_1(\chi_u)} h(v'_{p', q'} * v_{i'+jN_u, q}) = \\ &= \delta_{v', v} \delta_{q', q} \frac{f'_{-1}(u_{p-jN_u, i})}{f'_1(\chi_u)} \frac{f_{-1}(v_{i'+jN_u, p'})}{f_1(\chi_v)}. \end{aligned}$$

Exchanging the roles of x and x' gives

$$\begin{aligned} (E(x'), x) &= \overline{(x, E(x'))} = \\ &= \delta_{v, v'} \delta_{q, q'} \frac{\overline{f'_{-1}(u_{p'-j'N_u, i'})}}{f'_1(\chi_u)} \frac{\overline{f_{-1}(v_{i'+j'N_u, p'})}}{f_1(\chi_{v'})} = \\ &= \delta_{v, v'} \delta_{q, q'} \frac{f'_{-1}(u_{i', p'-j'N_u, i})}{f'_1(\chi_u)} \frac{f_{-1}(v_{p, i+j'N_u, p'})}{f_1(\chi_{v'})}, \end{aligned}$$

by the relation $\overline{f_{-1}(u_{r,s})} = f_{-1}(u_{s,r})$ shown in [10], (5.17)–(5.18). It suffices to assume then $v = v'$ and $q = q'$, so we also have $j = j'$. The previous lemma now completes the proof of (5.1).

Finally, (5.1) shows that $x - E(x)$ and $E(x)$ are orthogonal to each other since $E^2 = E$ by Lemma 5.1. Therefore in the Hilbert space norm: $\|E(x)\|^2 \leq \|x\|^2$. The rest is now clear.

A representation v of G on a vector space V is a linear map $v : V \rightarrow V \odot \mathcal{A}_\infty$ such that

$$v \otimes \iota_{\mathcal{A}} \circ v = \iota_V \otimes \Delta \circ v, \quad (5.3)$$

$$v(V) \mathcal{A}_\infty = V \odot \mathcal{A}_\infty. \quad (5.4)$$

In the last equation $V \odot \mathcal{A}_\infty$ is regarded as a right \mathcal{A}_∞ -module in the natural way. Note that a unitary finite dimensional representation of G is a vector space representation of G . Moreover, for any finite dimensional Hilbert space H ,

$$\iota_H \otimes \Delta : H \odot \mathcal{A}_\infty \rightarrow H \odot \mathcal{A}_\infty \odot \mathcal{A}_\infty$$

is vector space representation of G on $H \odot \mathcal{A}_\infty$.

If v and v' are two vector space representations of G on V and V' respectively, a linear map $T : V \rightarrow V'$ is called an intertwiner if

$$T \otimes 1_{\mathcal{A}} \circ v = v' \circ T.$$

The space of all such intertwiners will be denoted by (v, v') .

We are now ready to define induced G -representations on the spaces $X_{\text{Ind}_\infty(u)}$ and $H_{\text{Ind}(u)}$.

5.4 Proposition *Let u be a unitary finite dimensional representation of a compact quantum subgroup K of a compact quantum group G .*

- a) Then the idempotent $E : H \otimes \mathcal{A}_\infty \rightarrow X_{\text{Ind}_\infty(u)}$ intertwines the vector space G -representation

$$\iota_H \otimes \Delta : H \odot \mathcal{A}_\infty \rightarrow H \odot \mathcal{A}_\infty \odot \mathcal{A}_\infty$$

with itself. Therefore the restriction of $\iota_H \otimes \Delta$ to $X_{\text{Ind}_\infty(u)}$ gives rise to a vector space representation

$$\text{Ind}_\infty(u) : X_{\text{Ind}_\infty(u)} \rightarrow X_{\text{Ind}_\infty(u)} \odot \mathcal{A}_\infty.$$

- b) The same map $\iota_H \otimes \Delta$ restricted to the right Hilbert module $X_{\text{Ind}_0(u)}$ is a bounded linear map

$$\text{Ind}_0(u) : X_{\text{Ind}_0(u)} \rightarrow X_{\text{Ind}_0(u)} \otimes \mathcal{A},$$

where \otimes denotes the exterior tensor product.

- c) The map $\text{Ind}_0(u)$ extends uniquely to a map

$$\text{Ind}(u) : H_{\text{Ind}(u)} \rightarrow H_{\text{Ind}(u)} \otimes \mathcal{A}$$

satisfying the assumptions of Prop. 3.1, and therefore a unitary G -representation.

Proof a) $H \odot \mathcal{A}_\infty$ is linearly spanned by simple tensors of the form $x = \psi_i \otimes v_{p,q}$, with $\{\psi_r\}$ an orthonormal basis of H and $v_{p,q}$ coefficients of an irreducible unitary representation of G . By (5.2)

$$\begin{aligned} \iota_H \otimes \Delta \circ E(x) &= \sum_{r,k} h'(u_{i,r}^* \pi(v_{p,k})) \psi_r \otimes \Delta(v_{k,q}) = \\ \sum_{r,k,s} h'(u_{i,r}^* \pi(v_{p,k})) \psi_r \otimes v_{k,s} \otimes v_{s,q} &= \sum_s E \otimes \iota(\psi_i \otimes v_{p,s} \otimes v_{s,q}) = \\ E \otimes \iota \circ \Delta(x). \end{aligned}$$

Therefore the restriction $\text{Ind}_\infty(u)$ of $\iota_H \otimes \Delta$ to $X_{\text{Ind}_\infty(u)}$ takes that subspace into $X_{\text{Ind}_\infty(u)} \odot \mathcal{A}_\infty$ and clearly satisfies relation (5.3). We show (5.4) with $V = X_{\text{Ind}_\infty(u)} = E(H \otimes \mathcal{A}_\infty)$.

$$\begin{aligned} \text{Ind}_\infty(u)(E(H \otimes \mathcal{A}_\infty))\mathcal{A}_\infty &= E \otimes \iota(\iota_H \otimes \Delta(H \otimes \mathcal{A}_\infty))\mathcal{A}_\infty = \\ E \otimes \iota(H \otimes (\Delta(\mathcal{A}_\infty)I \otimes \mathcal{A}_\infty)) &= E \otimes \iota(H \otimes \mathcal{A}_\infty \odot \mathcal{A}_\infty) = \\ X_{\text{Ind}_\infty(u)} \odot \mathcal{A}_\infty. \end{aligned}$$

b) Obviously $\iota_H \otimes \Delta$ is a bounded linear map from the right Hilbert module $H \otimes \mathcal{A}$ to the exterior tensor product of right Hilbert modules $(H \otimes \mathcal{A}) \otimes \mathcal{A}$. The space $X_{\text{Ind}_0(u)}$ is a norm closed subspace of $H \otimes \mathcal{A}$. The following computations show that $\iota_H \otimes \Delta(X_{\text{Ind}_0(u)}) \subset X_{\text{Ind}_0(u)} \otimes \mathcal{A}$. We show that for any $T \in X_{\text{Ind}_0(u)}$,

$$u \otimes \iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}}(\iota_H \otimes \Delta(T)) = \iota_H \otimes \delta'_K \otimes \iota_{\mathcal{A}}(\iota_H \otimes \Delta(T)).$$

The left hand side equals

$$\begin{aligned} \iota_H \otimes \iota_{\mathcal{A}'} \otimes \Delta(u \otimes \iota_{\mathcal{A}}(T)) &= \iota_H \otimes \iota_{\mathcal{A}'} \otimes \Delta(\iota_H \otimes \delta'_K(T)) = \\ \iota_H \otimes \iota_{\mathcal{A}'} \otimes \Delta \circ \iota_H \otimes \pi \otimes \iota_{\mathcal{A}} \circ \iota_H \otimes \Delta(T) &= \iota_H \otimes \pi \otimes \iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}} \circ \iota_H \otimes \Delta \otimes \iota_{\mathcal{A}} \circ \iota_H \otimes \Delta(T) = \\ \iota_H \otimes \delta'_K \otimes \iota_{\mathcal{A}}(\iota_H \otimes \Delta(T)). \end{aligned}$$

Since the norm on the range space coincides with the norm inherited from $H \otimes \mathcal{A} \otimes \mathcal{A}$, and since $\iota_H \otimes \Delta : H \otimes \mathcal{A} \rightarrow H \otimes \mathcal{A} \otimes \mathcal{A}$ is bounded, $\text{Ind}_0(u)$ is bounded as well.

c) We show relation (3.1). For $\psi, \psi' \in H$, $a, a' \in \mathcal{A}$, $T = \psi \otimes a$, $T' = \psi' \otimes a'$,
 $(\iota_H \otimes \Delta(T), \iota_H \otimes \Delta(T'))_{\mathcal{A}} = (\psi, \psi')h \otimes \iota(\Delta(a^* a')) = (\psi, \psi')h(a^* a') = (T, T')I$,

so the relation holds a fortiori on $X_{\text{Ind}_0(u)}$, and $\iota_H \otimes \Delta$ extends on the completion $H_{\text{Ind}(u)}$ to a map satisfying the same relation. By a), $\text{Ind}(u)(X_{\text{Ind}_\infty(u)})\mathcal{A}_\infty = X_{\text{Ind}_\infty(u)} \odot \mathcal{A}_\infty$, and this subspace is norm dense in the right Hilbert \mathcal{A} -module $H_{\text{Ind}(u)} \otimes \mathcal{A}$.

We shall call $\text{Ind}(u)$ the *representation induced* from u . We conclude this section with a result on Frobenius reciprocity for induced representations.

5.5 Theorem *Let K be a compact quantum subgroup of the compact quantum group G . Let u and v be finite dimensional unitary representations of K and G respectively, with v faithful. Then the spaces $(v, \text{Ind}_\infty(u))$ and $(v \upharpoonright_K, u)$ are linearly isomorphic.*

Proof Let H_u and H_v denote the spaces of u and v respectively. For $T \in (v, \text{Ind}_\infty(u))$ and $\psi \in H_v$, $\iota \otimes e(T(\psi))$, with e the counit of \mathcal{A}_∞ [12], is an element of H_u . So we get a linear map S from H_v to H_u . We show that this map is an intertwiner between the desired representations.

$$\begin{aligned} u \circ S(\psi) &= u(\iota \otimes e(T(\psi))) = \iota_{H_u} \otimes \iota_{\mathcal{A}'} \otimes e(u \otimes \iota_{\mathcal{A}}(T\psi)) = \\ \iota_{H_u} \otimes \iota_{\mathcal{A}'} \otimes e(\iota_{H_u} \otimes \delta'_K(T\psi)) &= \iota_{H_u} \otimes \iota_{\mathcal{A}'} \otimes e(\iota_{H_u} \otimes \pi \otimes \iota_{\mathcal{A}} \circ \iota_{H_u} \otimes \Delta(T\psi)) = \\ \iota_{H_u} \otimes \pi \circ \iota_{H_u} \otimes \iota_{\mathcal{A}} \otimes e \circ \iota_{H_u} \otimes \Delta(T\psi) &= \iota_{H_u} \otimes \pi(T\psi). \end{aligned}$$

On the other hand:

$$\begin{aligned} S \otimes \iota_{\mathcal{A}'} v \upharpoonright_K (\psi) &= \iota_{H_u} \otimes e \otimes \iota_{\mathcal{A}'} \circ T \otimes \iota_{\mathcal{A}'} \circ \iota_{H_v} \otimes \pi(v(\psi)) = \\ \iota_{H_u} \otimes e \otimes \pi(T \otimes \iota_{\mathcal{A}}(v(\psi))) &= \iota_{H_u} \otimes e \otimes \pi \circ \iota_{H_u} \otimes \Delta(T(\psi)) = \\ \iota_{H_u} \otimes \pi \circ \iota_{H_u} \otimes (e \otimes \iota_{\mathcal{A}} \circ \Delta)T(\psi) &= \iota_{H_u} \otimes \pi(T(\psi)). \end{aligned}$$

Let now start from an operator $S \in (v \upharpoonright_K, u)$. For a vector $\psi \in H_v$ we set

$$T(\psi) := S \otimes \iota_{\mathcal{A}}(v(\psi)) \in H_u \odot \mathcal{A}_\infty.$$

We show that $T(\psi)$ lies in the space of $\text{Ind}_\infty(u)$.

$$u \otimes \iota_{\mathcal{A}}(S \otimes \iota_{\mathcal{A}}(v(\psi))) = (uS) \otimes \iota_{\mathcal{A}}(v(\psi)) =$$

$$\begin{aligned}
(S \otimes \iota_{A'} \circ v \upharpoonright_K) \otimes \iota_A(v(\psi)) &= S \otimes \iota_{A'} \otimes \iota_A \circ \iota_{H_v} \otimes \pi \otimes \iota_A \circ v \otimes \iota_A(v(\psi)) = \\
S \otimes \iota_{A'} \otimes \iota_A \circ \iota_{H_v} \otimes \pi \otimes \iota_A \circ \iota_{H_v} \otimes \Delta(v(\psi)) &= S \otimes \iota_{A'} \otimes \iota_A \circ \iota_{H_v} \otimes \delta'_K(v(\psi)) = \\
&\iota_{H_u} \otimes \delta'_K(S \otimes \iota_A(v(\psi))).
\end{aligned}$$

We check that $T \in (v, \text{Ind}_\infty(u))$.

$$\begin{aligned}
\iota_{H_u} \otimes \Delta T(\psi) &= \iota_{H_u} \otimes \Delta(S \otimes \iota_A(v(\psi))) = \\
S \otimes \iota_A \iota_A(\iota_{H_v} \otimes \Delta(v(\psi))) &= S \otimes \iota_A \otimes \iota_A(v \otimes \iota_A(v(\psi))) = T \otimes \iota_A(v(\psi)).
\end{aligned}$$

Finally we check that the maps $T \rightarrow S$ and $S \rightarrow T$ are inverses of one another.

$$\begin{aligned}
(\iota_{H_u} \otimes e \circ T) \otimes \iota_A(v(\psi)) &= \iota_{H_u} \otimes e \otimes \iota_A \circ \iota_{H_u} \otimes \Delta(T\psi) = T(\psi), \\
\iota_{H_u} \otimes e(S \otimes \iota_A v(\psi)) &= S \otimes e(v(\psi)) = S(\psi),
\end{aligned}$$

since v is faithful.

6 Ergodicity and transitivity

It is well known that transitivity characterizes G -actions arising from compact subgroups defined up to conjugacy, in the following way. Let X be a compact topological space on which a compact group G acts continuously by homeomorphisms on the right. If the G -action is transitive, the stabilizer of a point $x \in X$ is a closed subgroup G_x of G , and the map $xg \in X \rightarrow G_x g \in G_x \backslash G$ is a homeomorphism. Stabilizers of different points are conjugate closed subgroups of G . Conversely, given a closed subgroup K of G , the quotient space $K \backslash G$ with its quotient topology becomes a quotient right G -space under the action $g : Kg_1 \rightarrow Kg_1g$. The following fact is well known.

6.1 Proposition *Let X be a compact right G -space over a compact group G . The following properties are equivalent:*

- a) *the automorphic action $\alpha : g \in G \rightarrow \alpha_g \in \text{Aut}(\mathcal{C}(X))$, with $\alpha_g(f)(x) = f(xg)$, is ergodic,*
- b) *the G -action on X by homeomorphisms is transitive,*
- c) *there is no proper closed subset $F \subset X$ such that $FG = F$.*

If one of the above properties holds then there is a closed subgroup K of G , determined up to conjugation, such that $(\mathcal{C}(X), \alpha)$ is isomorphic to the automorphic action on $\mathcal{C}(K \backslash G)$ induced by the natural right G -action on $K \backslash G$.

Let now G be a compact quantum group acting on a (possibly noncommutative) unital C^* -algebra \mathcal{C} . Is ergodicity still equivalent to some sort of topological transitivity of that action?

In the case where G is a classical compact group acting pointwise continuously in norm on a unital C^* -algebra \mathcal{C} , $\alpha : G \rightarrow \text{Aut}(\mathcal{C})$, the second question has been investigated by Longo and Peligrad in [4], who proved the following theorem.

Recall that a subset M of \mathcal{C} is called G -invariant if $\alpha_g(M) \subset M$ for all $g \in G$. Clearly, since each g has an inverse g^{-1} , $\alpha_g(M) = M$ for all $g \in G$.

6.2 Theorem [4] *The following properties are equivalent:*

- a) $\mathcal{C}^\alpha = \mathbb{C}I$,
- b) *there is no proper, closed, G -invariant left ideal of \mathcal{C} ,*
- c) *there is no proper, hereditary, G -invariant C^* -subalgebra of \mathcal{C} .*

We give a generalization of the above result in the quantum group case. A C^* -subalgebra, or a closed left ideal, \mathcal{B} of \mathcal{C} will be called G -invariant if $\delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{A}$.

6.3 Theorem *Let $\delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{A}$ an action of the compact quantum group $G = (\mathcal{A}, \Delta)$ on the unital C^* -algebra \mathcal{C} such that $I \otimes \mathcal{A}\delta(\mathcal{C}) = \mathcal{C} \otimes \mathcal{A}$. Then the following properties are equivalent:*

- a) $\mathcal{C}^\delta = \mathbb{C}I$,
- b) *there is no proper, closed G -invariant, left ideal \mathcal{J} of \mathcal{C} such that $I \otimes \mathcal{A}\delta(\mathcal{J})$ is norm dense in $\mathcal{J} \otimes \mathcal{A}$,*
- c) *there is no proper, G -invariant, hereditary C^* -subalgebra \mathcal{H} of \mathcal{C} such that the hereditary C^* -subalgebra of $\mathcal{C} \otimes \mathcal{A}$ generated by $\delta(\mathcal{H})$ is $\mathcal{H} \otimes \mathcal{A}$,*
- d) *there is no proper, open projection $p \in \mathcal{C}''$ such that $\delta''(p) = p \otimes I$.*

Proof Closed left ideals of \mathcal{C} are in bijective correspondence with hereditary C^* -subalgebras via the map assigning to the left ideal \mathcal{J} the algebra $\mathcal{H} = \mathcal{J} \cap \mathcal{J}^*$ with inverse the map assigning to the algebra \mathcal{H} the ideal $\mathcal{J} = \{j \in \mathcal{C} : j^*j \in \mathcal{H}\}$ (see [5]). If \mathcal{J} and \mathcal{H} are in correspondence, for any C^* -algebra \mathcal{A} , $\mathcal{J} \otimes \mathcal{A}$ and $\mathcal{H} \otimes \mathcal{A}$ are a closed left ideal and a hereditary C^* -subalgebra of $\mathcal{C} \otimes \mathcal{A}$ in correspondence as well. If \mathcal{J} is G -invariant and $h \in \mathcal{H}$ then $\delta(h) \in \mathcal{J} \otimes \mathcal{A} \cap \mathcal{J}^* \otimes \mathcal{A} = \mathcal{H} \otimes \mathcal{A}$, so \mathcal{H} is G -invariant. Conversely, if \mathcal{H} is G -invariant and $j \in \mathcal{J}$ then $\delta(j^*j) \in \mathcal{H} \otimes \mathcal{A}$. Therefore $\delta(j) \in \mathcal{J} \otimes \mathcal{A}$. Thus G -invariant closed left ideals are in bijective correspondence with G -invariant hereditary C^* -subalgebras. We claim that if \mathcal{J} is a closed left ideal of \mathcal{C} then the closure $\bar{\mathcal{J}}$ of $I \otimes \mathcal{A}\delta(\mathcal{J})$ is a left ideal of $\mathcal{C} \otimes \mathcal{A}$. We only have to show that $\mathcal{C} \otimes I\bar{\mathcal{J}} \subset \bar{\mathcal{J}}$. Now since $I \otimes \mathcal{A}\delta(\mathcal{C})$ is dense, $\mathcal{C} \otimes I$ is contained in the closure of $I \otimes \mathcal{A}\delta(\mathcal{C})$, so $\mathcal{C} \otimes I\bar{\mathcal{J}}$, which coincides with the closure of $I \otimes \mathcal{A}\mathcal{C} \otimes I\delta(\mathcal{J})$, is contained in the closure of $I \otimes \mathcal{A}\delta(\mathcal{C}\bar{\mathcal{J}})$, which is in turn contained in $\bar{\mathcal{J}}$. By [5], hereditary C^* -subalgebras, or closed left ideals, of a unital C^* -algebra \mathcal{B} are in bijective correspondence with open projections of \mathcal{B}'' . If p is the projection of \mathcal{C}'' corresponding to a closed, left, G -invariant

ideal \mathcal{J} of \mathcal{C} then $\delta''(p)$ must coincide with the open projection of $(\mathcal{C} \otimes \mathcal{A})''$ corresponding to \mathcal{J} , the closure of the ideal $I \otimes \mathcal{A}\delta(\mathcal{J})$; G -invariance of \mathcal{J} shows that this ideal is contained in $\mathcal{J} \otimes \mathcal{A}$, so $\delta''(p) \leq p \otimes I$. If in addition $I \otimes \mathcal{A}\delta(\mathcal{J})$ is norm dense in $\mathcal{J} \otimes \mathcal{A}$ then $p \otimes I\delta''(p) = p \otimes I$, so $\delta''(p) = p \otimes I$. Conversely, any such projection corresponds to a closed G -invariant left ideal \mathcal{J} such that $I \otimes \mathcal{A}\delta(\mathcal{J}) = \mathcal{J} \otimes \mathcal{A}$. Again, for a hereditary G -invariant C^* -subalgebra \mathcal{H} of \mathcal{C} , the hereditary C^* -subalgebra generated by $\delta(\mathcal{H})$ corresponds to the projection $\delta(p)$, which is in turn dominated by $p \otimes I$ by G -invariance. Requiring that it coincides with $\mathcal{H} \otimes \mathcal{A}$ fixes $\delta''(p) = p \otimes I$. We have thus proven the equivalence of b), c) and d). The implication b) \rightarrow a) is easy: if there were a positive non scalar element a in \mathcal{C}^δ then its spectrum would contain at least two points x_1 and x_2 . Let f be a continuous function on the spectrum such that $f(x_1) = 1$, $f(x_2) = 0$. The closed left ideal \mathcal{J} generated by $f(a)$ is then proper and G -invariant. The norm closure of $I \otimes \mathcal{A}\delta(\mathcal{J})$ is generated by $I \otimes \mathcal{A}\delta(\mathcal{C})f(a) \otimes I$, or by $\mathcal{C}f(a) \otimes \mathcal{A}$ since $I \otimes \mathcal{A}\delta(\mathcal{C})$ is dense in $\mathcal{C} \otimes \mathcal{A}$. We are left to show that a) \rightarrow d). Let p be an open projection of \mathcal{C}'' such that $\delta''(p) = p \otimes I$. Consider the conditional expectation $E : \mathcal{C} \rightarrow \mathcal{C}^\delta$ over the fixed points obtained overaging over G : $E(c) := \iota \otimes h \circ \delta(c)$. By universality of \mathcal{C}'' , E extends to a normal positive map $E'' : \mathcal{C}'' \rightarrow \mathcal{C}''$ such that $E''(c'') = c''$ whenever $\delta''(c'') = c'' \otimes I$. In particular, $E''(p) = p$. By 3.11.9 in [5] p can be obtained as a strong limit of a bounded monotone increasing net x_α from \mathcal{C}^+ , therefore $E''(p) = p$ is the strong limit of the monotone increasing net $E(x_\alpha)$. Since $\mathcal{C}^\delta = \mathbb{C}I$, p must be a multiple of the identity, i.e. either $p = 0$, $p = I$, and the proof is complete.

Under stronger assumptions on the quantum group G , conditions b) through d) in the previous theorem take a more relaxed form.

6.4 Theorem *If the Haar measure h and the action δ are faithful maps, then, under the same assumption as in the previous theorem, the following conditions are equivalent:*

- a) $\mathcal{C}^\delta = \mathbb{C}$,
- b') *there is no proper closed G -invariant left ideal of \mathcal{C} ,*
- c') *there is no proper hereditary G -invariant C^* -subalgebra of \mathcal{C} ,*
- d') *there is no proper open projection $p \in \mathcal{C}''$ such that $\delta(p) \leq p \otimes I$.*

Proof Indeed, projections as in d') are in one to one correspondence with ideals as in b') and algebras as in c'). If $p \in \mathcal{C}''$ is a nonzero projection satisfying the condition stated in d') then $E''(p) \leq p$ and p is the strong limit of an increasing net x_α from \mathcal{C}^+ with nonzero elements, so $E''(p)$ turns out to be a strong limit of the increasing net $E(x_\alpha)$, with nonzero entries. If the action is ergodic then $E''(p)$ is a nonzero scalar, and therefore $p = I$.

7 Ergodicity and quotient spaces

When is an ergodic action $\delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{A}$ of a compact quantum group $G = (\mathcal{A}, \Delta)$ on a C^* -algebra, isomorphic to an action on some quantum quotient space $K \backslash G = (\mathcal{A}^{\delta_K}, \eta')$?

Ergodicity is clearly not enough. Just think of the case where G is a group. The question has a negative answer if \mathcal{C} is not commutative, and positive if \mathcal{C} is commutative, thanks to Prop. 6.1.

But, of course, commutativity of \mathcal{C} can not be assumed in the quantum case. We look for some other property, necessary also in the case where G is a quantum group. The following is a key observation.

7.1 Proposition *Let $\alpha : G \rightarrow \text{Aut}(\mathcal{C})$ be a strongly continuous, ergodic, action of a compact group on a unital C^* -algebra \mathcal{C} . Then the following conditions are equivalent:*

- a) \mathcal{C} has a character,
- b) \mathcal{C} is commutative.

If one of the above condition is satisfied, the given dynamical system is isomorphic to the system arising from a quotient right G -space.

Proof We need to show that a) \rightarrow b). Let χ be a character of \mathcal{C} . The stabilizer of χ , $G_\chi := \{g \in G : \chi \circ \alpha_g = \chi\}$, is a closed subgroup of G , and the map

$$\rho : c \in \mathcal{C} \rightarrow (g \in G \rightarrow \chi \circ \alpha_g(c)) \in \mathcal{C}(G)$$

is a $*$ -homomorphism with range included in the commutative C^* -algebra $\mathcal{C}(G_\chi \backslash G)$. This map intertwines the corresponding automorphic G -actions. The two sided closed ideal of \mathcal{C} : $\mathcal{J} = \{c \in \mathcal{C} : \chi(\alpha_g(c)) = 0, g \in G\}$ is obviously G -invariant and does not contain the identity, so by ergodicity, $\mathcal{J} = 0$. It follows that the $*$ -homomorphism $\mathcal{C} \rightarrow \mathcal{C}(G)$ assigning to an element $c \in \mathcal{C}$ the continuous function $g \in G \rightarrow \chi(\alpha_g(c))$, is faithful. Thus \mathcal{C} is commutative, and G acts transitively on the spectrum of \mathcal{C} . In particular, the closed orbit $\{\chi \circ \alpha_g, g \in G\}$ must coincide with whole spectrum of \mathcal{C} . A straightforward application of the Stone-Weierstrass theorem shows that ρ must be surjective.

How many quantum coset spaces with a $*$ -character do there exist? Think of the following construction. A quantum coset space is the fixed point algebra of the Hopf C^* -algebra \mathcal{A} of a quantum group $G = (\mathcal{A}, \Delta)$ under the action of a subgroup (see Sect. 4). Therefore it suffices to look for a $*$ -character on \mathcal{A} .

On the other hand Woronowicz shows in [10] that every compact matrix pseudogroup has a densely defined $*$ -character e : the counit. This is a $*$ -homomorphism $e : \mathcal{A} \rightarrow \mathbb{C}$, defined only on the smooth part \mathcal{A}_∞ of \mathcal{A} , such that for $a \in \mathcal{A}_\infty$,

$$\iota \otimes e \circ \Delta(a) = a.$$

But, he also shows in [11] that one can obtain compact quantum groups from any suitable category of finite dimensional Hilbert spaces, via Tannaka-Krein

duality. In fact these groups obtained from categories are completion of their smooth part with respect to the maximal C^* -norm. Therefore for these groups the counit must be an everywhere defined $*$ -character. For example, the group $S_\mu U(d)$ has such a character.

In conclusion, for sufficiently many quantum groups G , if a compact quantum subgroup K of G is given, the restriction e_K of the counit e to the right coset space \mathcal{A}^{δ_K} is a continuous $*$ -character of that C^* -subalgebra of \mathcal{A} .

We show a property possessed by the counit.

7.2 Lemma *If the action $\delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{A}$ of the compact quantum group G on \mathcal{A} is faithful, and if e is an everywhere defined counit of G then $\iota \otimes e \circ \delta(c) = c$ for all $c \in \mathcal{C}$.*

Proof It suffices to show that $\delta(\iota \otimes e \circ \delta(c)) = \delta(c)$. Indeed, the l.h.s. equals

$$\begin{aligned} \iota \otimes \iota \otimes e \circ \delta \otimes \iota \circ \delta(c) &= \iota \otimes \iota \otimes e \circ \iota \otimes \Delta \circ \delta(c) = \\ \iota \otimes (\iota \otimes e \circ \Delta)(\delta(c)) &= \delta(c). \end{aligned}$$

Consider an action δ of a compact quantum group $G = (\mathcal{A}, \Delta)$ on a unital C^* -algebra \mathcal{C} . If \mathcal{C} has a character χ , Prop. 7.1 suggests how to construct an intertwiner from the system $(\mathcal{C}, \delta, \chi)$ to some system of the form $K \setminus G = (\mathcal{A}^{\delta_K}, \eta', e_K)$.

7.3 Theorem *Let $\delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{A}$ be an action of G on a unital C^* -algebra \mathcal{C} . Let $\chi : \mathcal{C} \rightarrow \mathbb{C}$ be a character of \mathcal{C} . Then there is a compact quantum subgroup $G_\chi = (\mathcal{A}', \Delta')$ of G such that the map*

$$T_\chi := \chi \otimes \iota_{\mathcal{A}} \circ \delta : \mathcal{C} \rightarrow \mathcal{A}^{\delta_{G_\chi}}$$

is a $$ -homomorphism intertwining the corresponding G -actions:*

$$\eta' \circ T_\chi = T_\chi \otimes \iota_{\mathcal{A}} \circ \delta.$$

Furthermore, if $\pi : \mathcal{A} \rightarrow \mathcal{A}'$ is the quotient map,

$$\pi(T_\chi(c)) = \chi(c)I_{\mathcal{A}'}, \quad c \in \mathcal{C}.$$

Also, if δ is faithful and if G has an everywhere defined counit e then

$$e(T_\chi(c)) = \chi(c), \quad c \in \mathcal{C}.$$

Proof Consider the linear subspace M_χ of \mathcal{A} generated by

$$\{\chi \otimes \iota \circ \delta(c) - \chi(c)I, c \in \mathcal{C}\}.$$

Notice that

$$\Delta(\chi \otimes \iota \circ \delta(c) - \chi(c)I) = (\chi \otimes \iota \circ \delta - \chi I) \otimes \iota(\delta(c)) + I \otimes (\chi \otimes \iota \circ \delta(c) - \chi(c)I),$$

so

$$\Delta(M_\chi) \subset M_\chi \otimes \mathcal{A} + I \otimes M_\chi.$$

Let J_χ be the closed ideal generated by M_χ in \mathcal{A} . This is a $*$ -ideal since M_χ is $*$ -invariant. The above relation shows that $\Delta(J_\chi)$ is contained in $J_\chi \otimes \mathcal{A} + \mathcal{A} \otimes J_\chi$. Set $\mathcal{A}' = \mathcal{A}/J_\chi$, and define $\Delta'([a]) = \pi \otimes \pi(\Delta(a))$, with $\pi : \mathcal{A} \rightarrow \mathcal{A}'$ the quotient map. This map is well defined and defines a nondegenerate coassociative coproduct on \mathcal{A}' . It is now obvious that $G_\chi := (\mathcal{A}', \Delta')$ is a compact subgroup of G , and, by definition,

$$\chi \otimes \pi \circ \delta(c) = \chi(c)I_{\mathcal{A}'}, \quad c \in \mathcal{C}.$$

We show that the range of T_χ is included in $\mathcal{A}^{\delta'_{G_\chi}}$. For $c \in \mathcal{C}$,

$$\begin{aligned} \delta'_{G_\chi}(T_\chi(c)) &= \pi \otimes \iota \circ \Delta \circ \chi \otimes \iota \circ \delta(c) = \\ &= \chi \otimes \pi \otimes \iota \circ \iota \otimes \Delta \circ \delta(c) = \chi \otimes \pi \otimes \iota \circ \delta \otimes \iota \circ \delta(c) = \\ &= [(\chi \otimes \pi \circ \delta) \otimes \iota] \circ \delta(c) = \chi(\cdot)I \otimes \iota \circ \delta(c) = I \otimes T_\chi(c). \end{aligned}$$

We finally show that T_χ intertwines the corresponding G -actions. For $c \in \mathcal{C}$,

$$\begin{aligned} \eta' \circ T_\chi(c) &= \Delta \circ \chi \otimes \iota \circ \delta(c) = \\ &= \chi \otimes \iota \otimes \iota \circ \iota \otimes \Delta \circ \delta(c) = \chi \otimes \iota \otimes \iota \circ \delta \otimes \iota \circ \delta(c) = \\ &= T_\chi \otimes \iota \circ \delta(c). \end{aligned}$$

Finally, by the previous lemma,

$$e(T_\chi(c)) = \chi \otimes e \circ \delta(c) = \chi(\iota \otimes e \circ \delta(c)) = \chi(c).$$

The subgroup G_χ constructed along the proof of the previous theorem will be called the *subgroup stabilizing χ* .

We can summarize the results of this and the previous section in the following theorem.

7.4 Theorem *Let $\delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{A}$ be an action of the compact quantum group $G = (\mathcal{A}, \Delta)$ on a unital C^* -algebra \mathcal{C} endowed with a $*$ -character χ . Assume that*

- a) $I \otimes \mathcal{A}\delta(\mathcal{C})$ is norm dense in $\mathcal{C} \otimes \mathcal{A}$,
- b) the action δ is ergodic: $\mathcal{C}^\delta = \mathbb{C}$.

Assume furthermore that the action δ and the Haar measure h of G are faithful maps. It follows that T_χ is faithful.

Proof We show that the kernel \mathcal{J} of T_χ is G -invariant. For $j \in \mathcal{J}$, by Theorem 7.3, $T_\chi \otimes \iota_{\mathcal{A}}(\delta(j)) = \eta'(T_\chi(j)) = 0$, so $\delta(j) \in \ker(T_\chi \otimes \iota_{\mathcal{A}}) = \mathcal{J} \otimes \mathcal{A}$. Since $\mathcal{J} \neq \mathcal{C}$, we must have $\mathcal{J} = 0$ thanks to Theorem 6.4.

7.5 Corollary *If $\delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{A}$ is a faithful ergodic action of a compact quantum group $G = (\mathcal{A}, \Delta)$, with faithful Haar measure, on a commutative unital C^* -algebra \mathcal{C} satisfying the nondegeneracy property a) above, then (\mathcal{C}, δ) can be embedded faithfully into a quotient space of G .*

Remark 6 If we drop the assumption that h and δ are faithful maps, but if we know a priori that T_χ is surjective, then T_χ must be faithful, and therefore a $*$ -isomorphism. The reason is explained in the following lemma, which we include for future reference.

7.6 Lemma *Let $\delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{A}$ and $\eta : \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{A}$ be unital actions of the compact quantum group $G = (\mathcal{A}, \Delta)$, and let $T : \mathcal{C} \rightarrow \mathcal{D}$ be a surjective $*$ -homomorphism such that*

$$\eta \circ T = T \otimes \iota_{\mathcal{A}} \circ \delta.$$

If $I \otimes \mathcal{A}\delta(\mathcal{C})$ and $I \otimes \mathcal{A}\eta(\mathcal{D})$ are norm dense in $\mathcal{C} \otimes \mathcal{A}$ and $\mathcal{D} \otimes \mathcal{A}$ respectively, then the kernel \mathcal{J} of T is a closed, two sided, G -invariant ideal of \mathcal{C} such that $I \otimes \mathcal{A}\delta(\mathcal{J})$ is norm dense in $\mathcal{J} \otimes \mathcal{A}$. In particular, if δ is ergodic, T must be faithful.

Proof As in the proof of the previous theorem, one can show that \mathcal{J} is G -invariant. By Theorem 6.3, we are left to show that $I \otimes \mathcal{A}\delta(\mathcal{J})$ is dense in $\mathcal{J} \otimes \mathcal{A}$. Extend T to a normal $*$ -homomorphism $T'' : \mathcal{C}'' \rightarrow \mathcal{D}''$, and let p be the central projection of \mathcal{C}'' such that $\ker(T'') = \mathcal{C}''(I - p)$. $I - p$ is the open projection of \mathcal{C}'' corresponding to \mathcal{J} . We need to show, by Theorem 6.3, that $\delta''(I - p) = (I - p) \otimes I$. The restriction T_p of T'' to $\mathcal{C}''p$ is a normal $*$ -monomorphism with range \mathcal{D}'' . Since $I \otimes \mathcal{A}\eta(\mathcal{D})$ is norm dense in $\mathcal{D} \otimes \mathcal{A}$, it is a fortiori weakly dense in the von Neumann tensor product $\mathcal{D}'' \otimes \mathcal{A}''$. Pulling back this relation with $T_p^{-1} \otimes I$ shows that $I \otimes \mathcal{A}''\delta''(\mathcal{C}''p)$ is weakly dense in the von Neumann tensor product $\mathcal{C}''p \otimes \mathcal{A}''$. Now the property that $I \otimes \mathcal{A}\delta(\mathcal{C})$ is norm dense shows that the weak closure of $I \otimes \mathcal{A}''\delta''(\mathcal{C}''p)$ is a weakly closed left (and right) ideal of $\mathcal{C}'' \otimes \mathcal{A}''$ defined by the projection $\delta''(p)$. The density statement shows that $\delta''(p) = p \otimes I$, and therefore $\delta''(I - p) = (I - p) \otimes I$.

Remark 7 We conclude the paper noting that one can not expect in general an isomorphism of an ergodic G -space (\mathcal{C}, δ) with a quotient G -space $K \backslash G$ by a stabilizer subgroup. In fact, Wang shows in [9] an example of a compact quantum group acting ergodically on a commutative C^* -algebra for which the quotient space by a point stabilizer subgroup is not commutative.

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